

# The Role of Surface Tension for the Equation of State of Quark-Gluon Bags

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## Abstract

The temperature and chemical potential dependent surface tension of bags is introduced into the gas of quark-gluon bags model. The suggested model is solved analytically. It resolves a long standing problem of a unified description of the first and second order phase transition with the cross-over. Such an approach is necessary to model the complicated properties of quark-gluon plasma and hadronic matter from the first principles of statistical mechanics. In addition to the deconfinement phase transition, we found that at the curve of a zero surface tension coefficient there must exist the surface induced phase transition of the 2<sup>nd</sup> or higher order, which separates the pure quark gluon plasma (QGP) from the cross-over states. Thus, the present model predicts that the critical endpoint of quantum chromodynamics is the tricritical endpoint.

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## 1 Introduction

The strongly interacting matter properties studied in relativistic nuclear collisions has reached the stage when the predictions of the lattice quantum chromodynamics (QCD) can be checked experimentally on the existing data and future measurements at BNL RHIC, CERN SPS, and GSI FAIR. However, a comparison of the theoretical results with the experimental data is not straightforward because during the collision process the matter can have several phase transformations which are difficult to model. The latter reason stimulated the development of a wide range of phenomenological models of the strongly interacting matter equation of state which are used in dynamical simulations.

One of these models, the gas of bags model (GBM) [1, 2, 3], itself contains two well-known models of deconfined and confined phases: the bag model of QGP [5] and the hadron gas model [6]. Hence there were hopes [7] that an exact analytical solution of the GBM found in [2] could be helpful in understanding the properties of strongly interacting matter. However, this solution does not allow one to introduce the critical end point of the strongly interacting matter phase diagram. Also, a complicated construction of the line, along which the phase transition order gradually increases, suggested in [7], does look too artificial. Therefore, the present GBM formulation lacks an important physical input and is interesting only as a toy example which can be solved analytically. However, there are the great demands [8, 9, 10] for the phenomenological models, which can correctly describe the properties of the end point of the 1<sup>st</sup> order deconfinement phase transition (PT) to QGP.

In statistical mechanics there are several exactly solvable cluster models with the 1<sup>st</sup> order PT which describe the critical point properties very well. These models are built on the assumptions that the difference of the bulk part (or the volume dependent part) of free energy of two phases disappears at phase equilibrium and that, in addition, the difference of the surface part (or the surface tension) of free energy vanishes at the critical point. The most famous of them is the Fisher droplet model (FDM) [12, 13, 14] which has been successfully used to analyze the condensation of a gaseous phase (droplets of all sizes) into a liquid. The FDM has been applied to many different systems [13, 14].

The other well established statistical model, the statistical multifragmentation model (SMM) [15, 16, 17], was recently solved analytically both for infinite [18, 20] and for finite [21, 22] volumes of the system. In the SMM the surface tension temperature dependence differs from that one of the FDM, but it was shown [20] that the value of

Fisher exponent  $\tau_{SMM} = 1.825 \pm 0.025$ , which contradicts to the FDM value  $\tau_{FDM} \approx 2.16$ , but is consistent with ISiS Collaboration data [23] and EOS Collaboration data [24].

From the structure of these models, it follows that the GMB can be drastically improved by the inclusion of such a vitally important element as the surface tension of the quark-gluon bags. The obtained model is called the QGBST model. Its detailed discussion and the full list of related references can be found in [11, 25, 26].

The great success of the SMM initiated the studies of the surface partitions of large clusters within the Hills and Dales Model [27, 28] and led to a discovery of the origin of the temperature independent surface entropy similar to the FDM. It was proven that the surface tension coefficient of large clusters consisting of the discrete constituents should linearly depend on the temperature of the system [27] and must vanish at the critical endpoint. Thus, the Hills and Dales Model [27, 28] is our main guide in formulating the QGBST model. However, for definiteness we assume a certain dependence of the surface tension coefficient on temperature and baryonic chemical potential, and concentrate on the impact of surface tension of the quark-gluon bags on the properties of the deconfinement phase diagram and the QCD critical endpoint.

Here we will show that the existence of a cross-over at low values of the baryonic chemical potential along with the 1<sup>st</sup> order deconfinement PT at high baryonic chemical potentials leads to the existence of an additional PT of the 2<sup>nd</sup> or higher order along the curve where the surface tension coefficient vanishes [25]. Thus, it turns out that the QGBST model predicts the existence of the tricritical rather than critical endpoint.

The work is organized as follows. Sect. 2 contains the formulation of the QGBST model and analyze all possible singularities of its isobaric partition for vanishing baryonic densities. This analysis is generalized to non-zero baryonic densities in Sect.3. Sect. 4 is devoted to the analysis of the surface tension induced PT which exists above the deconfinement PT. The conclusions and research perspectives are summarized in Sect. 5.

## 2 The Role of Surface Tension

I begin with the isobaric partition:

$$\hat{Z}(s, T) \equiv \int_0^\infty dV \exp(-sV) Z(V, T) = \frac{1}{[s - F(s, T)]} \quad (1)$$

where the function  $F(s, T)$  is defined as follows

$$F(s, T) \equiv F_H(s, T) + F_Q(s, T) = \sum_{j=1}^n g_j e^{-v_j s} \phi(T, m_j) + u(T) \int_{V_0}^\infty dv \frac{\exp[-v(s - s_Q(T))]}{v^\tau}. \quad (2)$$

At the moment the particular choice of function  $F_Q(s, T)$  in (2) is not important. The key point of my treatment is that it should have the form of Eq. (2) which has a singularity at  $s = s_Q^*$  because for  $s < s_Q$  the integral over the bag volume  $v$  diverges at its upper limit. As will be shown below the isobaric partition (1) has two kind of singularities: the simple pole  $s = s_H^*$  and the essential singularity  $s = s_Q$ . The rightmost singularity defines the phase in which matter exists, whereas a PT occurs when two singularities coincide [2, 18, 25]. All singularities are defined by the equation

$$s^* = F(s^*, T), \quad (3)$$

Note that the exponential in (2) is nothing else, but a difference of the bulk free energy of a bag of volume  $v$ , i.e.  $-Tsv$ , which is under external pressure  $Ts$ , and the bulk free energy of the same bag filled with QGP, i.e.  $-Ts_Q v$ . At phase equilibrium this difference of the bulk free energies vanishes. Despite all positive features, Eq. (2) lacks the surface part of free energy of bags, which will be called a surface energy hereafter. In addition to the difference of the bulk free energies the realistic statistical models which demonstrated their validity, the FDM [12] and SMM [15], have the contribution of the surface energy which plays an important role in defining the phase diagram structure [18, 22]. Therefore, I modify Eq. (2) by introducing the surface energy of the bags in a general fashion [20]:

$$F_Q(s, T) = u(T) \int_{V_0}^\infty dv \frac{\exp[(s_Q(T) - s)v - \sigma(T)v^\kappa]}{v^\tau}, \quad (4)$$

where the ratio of the temperature dependent surface tension coefficient to  $T$  (the reduced surface tension coefficient hereafter) which has the form  $\sigma(T) = \frac{\sigma_o}{T} \cdot \left[ \frac{T_{cep} - T}{T_{cep}} \right]^{2k+1}$  ( $k = 0, 1, 2, \dots$ ). Here  $\sigma_o > 0$  can be a smooth function of

the temperature, but for simplicity I fix it to be a constant. For  $k = 0$  the two terms in the surface (free) energy of a  $v$ -volume bag have a simple interpretation [12]: thus, the surface energy of such a bag is  $\sigma_0 v^\kappa$ , whereas the free energy, which comes from the surface entropy  $\sigma_0 T_{cep}^{-1} v^\kappa$ , is  $-T \sigma_0 T_{cep}^{-1} v^\kappa$ . Note that the surface entropy of a  $v$ -volume bag counts its degeneracy factor or the number of ways to make such a bag with all possible surfaces. This interpretation can be extended to  $k > 0$  on the basis of the Hills and Dales Model [27, 28].

In choosing such a simple surface energy parameterization we follow the original Fisher idea [12] which allows one to account for the surface energy by considering some mean bag of volume  $v$  and surface  $v^\kappa$ . The consideration of the general mass-volume-surface bag spectrum is reserved for the future investigation. The power  $\kappa < 1$  which describes the bag's effective surface is a constant which, in principle, can differ from the typical FDM and SMM value  $\frac{2}{3}$ . This is so because near the deconfinement PT region QGP has low density and, hence, like in the low density nuclear matter [35], the non-spherical bags (spaghetti-like or lasagna-like [35]) can be favorable (see a [25] and references therein). A similar idea of "polymerization" of gluonic quasiparticles was introduced recently [36].

The second essential difference with the FDM and SMM surface tension parameterization is that we do not require the vanishing of  $\sigma(T)$  above the CEP. As will be shown later, this is the most important assumption which, in contrast to the GBM, allows one to naturally describe the cross-over from hadron gas to QGP. Note that negative value of the reduced surface tension coefficient  $\sigma(T)$  above the CEP does not mean anything wrong. As we discussed above, the surface tension coefficient consists of energy and entropy parts which have opposite signs [12, 27, 28]. Therefore,  $\sigma(T) < 0$  does not mean that the surface energy changes the sign, but it rather means that the surface entropy, i.e. the logarithm of the degeneracy of bags of a fixed volume, simply exceeds their surface energy. In other words, the number of non-spherical bags of a fixed volume becomes so large that the Boltzmann exponent, which accounts for the energy "costs" of these bags, cannot suppress them anymore.

Finally, the third essential difference with the FDM and SMM is that we assume that the surface tension in the QGBST model vanishes at some line in  $\mu_B - T$  plane, i.e.  $T_{cep} = T_{cep}(\mu_B)$ . However, in the subsequent sections we will consider  $T_{cep} = \text{Const}$  for simplicity, and in Sect. V we will discuss the necessary modifications of the model with  $T_{cep} = T_{cep}(\mu_B)$ .

The surface energy should, perhaps, be introduced into a discrete part of the mass-volume spectrum  $F_H$ , but a successful fitting of the particle yield ratios [6] with the experimentally determined hadronic spectrum  $F_H$  does not indicate such a necessity.

In principle, besides the bulk and surface parts of free energy, the spectrum (4) could include the curvature part as well, which may be important for small hadronic bubbles or for cosmological PT. We stress, however, that the curvature term has not been seen in such well established modes like the FDM, the SMM and many other systems [13, 14]. A special analysis of the free energy of 2- and 3-dimensional Ising clusters, using the Complement method [37], did not find any traces of the curvature term (see a detailed discussion in Ref. [25]).

According to the general theorem [2] the analysis of PT existence of the GCP is now reduced to the analysis of the rightmost singularity of the isobaric partition (1). Depending on the sign of the reduced surface tension coefficient, there are three possibilities.

(I) The first possibility corresponds to  $\sigma(T) > 0$ . Its treatment is very similar to the GBM choice (2) with  $\tau > 2$  [2]. In this case at low temperatures the QGP pressure  $Ts_Q(T)$  is negative and, therefore, the rightmost singularity is a simple pole of the isobaric partition  $s^* = s_H(T) = F(s_H(T), T) > s_Q(T)$ , which is mainly defined by a discrete part of the mass-volume spectrum  $F_H(s, T)$ . The last inequality provides the convergence of the volume integral in (4) (see the left panel in Fig. 1). On the other hand at very high  $T$  the QGP pressure dominates and, hence, the rightmost singularity is the essential singularity of the isobaric partition  $s^* = s_Q(T)$ . The phase transition occurs, when the singularities coincide:

$$s_H(T_c) \equiv \frac{p_H(T_c)}{T_c} = s_Q(T_c) \equiv \frac{p_Q(T_c)}{T_c}, \quad (5)$$

which is nothing else, but the Gibbs criterion. The graphical solution of Eq. (3) for all these possibilities is shown in Fig. 1. Like in the GBM [2, 7], the necessary condition for the PT existence is the finiteness of  $F_Q(s_Q(T), T)$  at  $s = s_Q(T)$ . It can be shown that the sufficient conditions are the following inequalities:  $F_Q(s_Q(T), T) > s_Q(T)$  for low temperatures and  $F(s_Q(T), T) < s_Q(T)$  for  $T \rightarrow \infty$ . These conditions provide that at low  $T$  the rightmost singularity of the isobaric partition is a simple pole, whereas for high  $T$  the essential singularity  $s_Q(T)$  becomes its rightmost one (see Fig. 1 and a detailed analysis of case  $\mu_B \neq 0$ ).

The PT order can be found from the  $T$ -derivatives of  $s_H(T)$ . Thus, differentiating (3) one finds

$$s'_H = \frac{G + u \mathcal{K}_{\tau-1}(\Delta, -\sigma) \cdot s'_Q}{1 + u \mathcal{K}_{\tau-1}(\Delta, -\sigma)}, \quad (6)$$

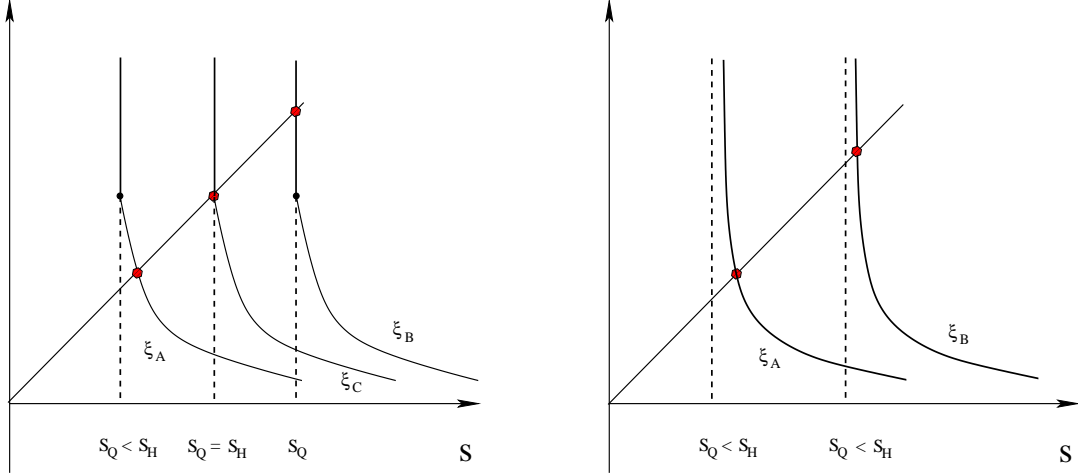


Figure 1: **Left panel.** Graphical solution of Eq. (3) which corresponds to a PT. The solution of Eq. (3) is shown by a filled hexagon. The function  $F(s, \xi)$  is shown by a solid curve for a few values of the parameter  $\xi$ . The function  $F(s, \xi)$  diverges for  $s < s_Q(\xi)$  (shown by dashed lines), but is finite at  $s = s_Q(\xi)$  (shown by black circle). At low values of the parameter  $\xi = \xi_A$ , which can be either  $T$  or  $\mu_B$ , the simple pole  $s_H$  is the rightmost singularity and it corresponds to hadronic phase. For  $\xi = \xi_B \gg \xi_A$  the rightmost singularity is an essential singularity  $s = s_Q(\xi_B)$ , which describes QGP. At intermediate value  $\xi = \xi_C$  both singularities coincide  $s_H(\xi_C) = s_Q(\xi_C)$  and this condition is a Gibbs criterion.

**Right panel.** Graphical solution of Eq. (3) which corresponds to a cross-over. The notations are the same as in the left panel. Now the function  $F(s, \xi)$  diverges at  $s = s_Q(\xi)$  (shown by dashed lines). In this case the simple pole  $s_H$  is the rightmost singularity for any value of  $\xi$ .

where the functions  $G$  and  $\mathcal{K}_{\tau-a}(\Delta, -\sigma)$  are defined as

$$G \equiv F'_H + \frac{u'}{u} F_Q + \frac{(T_{cep}-2kT)\sigma(T)}{(T_{cep}-T)T} u \mathcal{K}_{\tau-\kappa}(\Delta, -\sigma), \quad (7)$$

$$\mathcal{K}_{\tau-a}(\Delta, -\sigma) \equiv \int_{V_o}^{\infty} dv \frac{\exp[-\frac{\Delta v - \sigma(T)v^\kappa}{v^{\tau-a}}]}{v^{\tau-a}}, \quad (8)$$

where  $\Delta \equiv s_H - s_Q$ .

Now it is easy to see that the transition is of the 1<sup>st</sup> order, i.e.  $s'_Q(T_c) > s'_H(T_c)$ , provided  $\sigma(T) > 0$  for any  $\tau$ . The 2<sup>nd</sup> or higher order phase transition takes place provided  $s'_Q(T_c) = s'_H(T_c)$  at  $T = T_c$ . The latter condition is satisfied when  $\mathcal{K}_{\tau-1}$  diverges to infinity at  $T \rightarrow (T_c - 0)$ , i.e. for  $T$  approaching  $T_c$  from below. Like for the GBM choice (2), such a situation can exist for  $\sigma(T_c) = 0$  and  $\frac{3}{2} < \tau \leq 2$  [25]. Studying the higher  $T$ -derivatives of  $s_H(T)$  at  $T_c$ , one can find a more general statement, but for our purpose it is not necessary.

(II) The second possibility,  $\sigma(T) \equiv 0$ , described in the preceding paragraph, does not give anything new compared to the GBM [2, 7]. If the PT exists, then the graphical picture of singularities is basically similar to the left panel of Fig. 1. The only difference is that, depending on the PT order, the derivatives of  $F(s, T)$  function with respect to  $s$  should diverge at  $s = s_Q(T_c)$ .

(III) A principally new possibility exists for  $T > T_{cep}$ , where  $\sigma(T) < 0$ . In this case there exists a cross-over, if for  $T \leq T_{cep}$  the rightmost singularity is  $s_H(T)$ , which corresponds to the leftmost curve in the right panel of Fig. 1. Under the latter, its existence can be shown as follows. Let us solve the equation for singularities (3) graphically (see the right panel of Fig. 1). For  $\sigma(T) < 0$  the function  $F_Q(s, T)$  diverges at  $s = s_Q(T)$ . On the other hand, the partial derivatives  $\frac{\partial F_H(s, T)}{\partial s} < 0$  and  $\frac{\partial F_Q(s, T)}{\partial s} < 0$  are always negative. Therefore, the function  $F(s, T) \equiv F_H(s, T) + F_Q(s, T)$  is a monotonically decreasing function of  $s$ , which vanishes at  $s \rightarrow \infty$ . Since the left hand side of Eq. (3) is a monotonically increasing function of  $s$ , then there can exist a single intersection  $s^*$  of  $s$  and  $F(s, T)$  functions. Moreover, for finite  $s_Q(T)$  values this intersection can occur on the right hand side of the point  $s = s_Q(T)$ , i.e.  $s^* > s_Q(T)$  (see the right panel of Fig. 1). Thus, in this case the essential singularity  $s = s_Q(T)$  can become the rightmost one for infinite temperature only. In other words, the pressure of the pure

QGP can be reached at infinite  $T$ , whereas for finite  $T$  the hadronic mass spectrum gives a non-zero contribution into all thermodynamic functions. Note that such a behavior is typical for the lattice QCD data at zero baryonic chemical potential [38].

It is clear that in terms of the present model a cross-over existence means a fast transition of energy or entropy density in a narrow  $T$  region from a dominance of the discrete mass-volume spectrum of light hadrons to a dominance of the continuous spectrum of heavy QGP bags. This is exactly the case for  $\sigma(T) < 0$  because in the right vicinity of the point  $s = s_Q(T)$  the function  $F(s, T)$  decreases very fast and then it gradually decreases as function of  $s$ -variable. Since,  $F_Q(s, T)$  changes fast from  $F(s, T) \sim F_Q(s, T) \sim s_Q(T)$  to  $F(s, T) \sim F_H(s, T) \sim s_H(T)$ , their  $s$ -derivatives should change fast as well. Now, recalling that the change from  $F(s, T) \sim F_Q(s, T)$  behavior to  $F(s, T) \sim F_H(s, T)$  in  $s$ -variable corresponds to the cooling of the system (see the right panel of Fig. 1), we conclude that there exists a narrow region of temperatures, where the  $T$  derivative of system pressure, i.e. the entropy density, drops down from  $\frac{\partial p}{\partial T} \sim s_Q(T) + T \frac{ds_Q(T)}{dT}$  to  $\frac{\partial p}{\partial T} \sim s_H(T) + T \frac{ds_H(T)}{dT}$  very fast compared to other regions of  $T$ , if system cools. If, however, in the vicinity of  $T = T_{cep} - 0$  the rightmost singularity is  $s_Q(T)$ , then for  $T > T_{cep}$  the situation is different and the cross-over does not exist. A detailed analysis of this situation is given in Sect. 4.

Note also that all these nice properties would vanish, if the reduced surface tension coefficient is zero or positive above  $T_{cep}$ . This is one of the crucial points of the present model which puts forward certain doubts about the vanishing of the reduced surface tension coefficient in the FDM [12] and SMM [15]. These doubts are also supported by the first principle results obtained by the Hills and Dales Model [27, 28], because the surface entropy simply counts the degeneracy of a cluster of a fixed volume and it does not physically affect the surface energy of this cluster.

### 3 Generalization to Non-Zero Baryonic Densities

The possibilities (I)-(III) discussed in the preceding section remain unchanged for non-zero baryonic numbers. The latter should be included into consideration to make our model more realistic. To keep the presentation simple, we do not account for strangeness. The inclusion of the baryonic charge of the quark-gluon bags does not change the two types of singularities of the isobaric partition (1) and the corresponding equation for them (3), but it leads to the following modifications of the  $F_H$  and  $F_Q$  functions:

$$F_H(s, T, \mu_B) = \sum_{j=1}^n g_j e^{\frac{b_j \mu_B}{T} - v_j s} \phi(T, m_j), \quad (9)$$

$$F_Q(s, T, \mu_B) = u(T, \mu_B) \int_{V_0}^{\infty} dv \frac{\exp[(s_Q(T, \mu_B) - s)v - \sigma(T)v^\kappa]}{v^\tau}. \quad (10)$$

Here the baryonic chemical potential is denoted as  $\mu_B$ , the baryonic charge of the  $j$ -th hadron in the discrete part of the spectrum is  $b_j$ . The continuous part of the spectrum,  $F_Q$  can be obtained from some spectrum  $\rho(m, v, b)$  in the spirit of Ref. [29, 26], but this will lead us away from the main subject.

The QGP pressure  $p_Q = Ts_Q(T, \mu_B)$  can be also chosen in several ways. Here we use the bag model pressure  $p_Q = \frac{\pi^2}{90} T^4 \left[ \frac{95}{2} + \frac{10}{\pi^2} \left( \frac{\mu_B}{T} \right)^2 + \frac{5}{9\pi^4} \left( \frac{\mu_B}{T} \right)^4 \right] - B$ , but the more complicated model pressures, even with the PT of other kind like the transition between the color superconducting QGP and the usual QGP, can be, in principle, used.

The sufficient conditions for a PT existence are

$$F((s_Q(T, \mu_B=0)+0), T, \mu_B=0) > s_Q(T, \mu_B=0), \quad (11)$$

$$F((s_Q(T, \mu_B)+0), T, \mu_B) < s_Q(T, \mu_B), \forall \mu_B > \mu_A. \quad (12)$$

The condition (11) provides that the simple pole singularity  $s^* = s_H(T, \mu_B=0)$  is the rightmost one at vanishing  $\mu_B = 0$  and given  $T$ , whereas the condition (12) ensures that  $s^* = s_Q(T, \mu_B)$  is the rightmost singularity of the isobaric partition for all values of the baryonic chemical potential above some positive constant  $\mu_A$ . This can be seen in Fig. 1 for  $\mu_B$  being a variable. Since  $F(s, T, \mu_B)$ , where it exists, is a continuous function of its parameters, one concludes that, if the conditions (11) and (12), are fulfilled, then at some chemical potential  $\mu_B^c(T)$  the both singularities should be equal. Thus, one arrives at the Gibbs criterion (5), but for two variables

$$s_H(T, \mu_B^c(T)) = s_Q(T, \mu_B^c(T)). \quad (13)$$

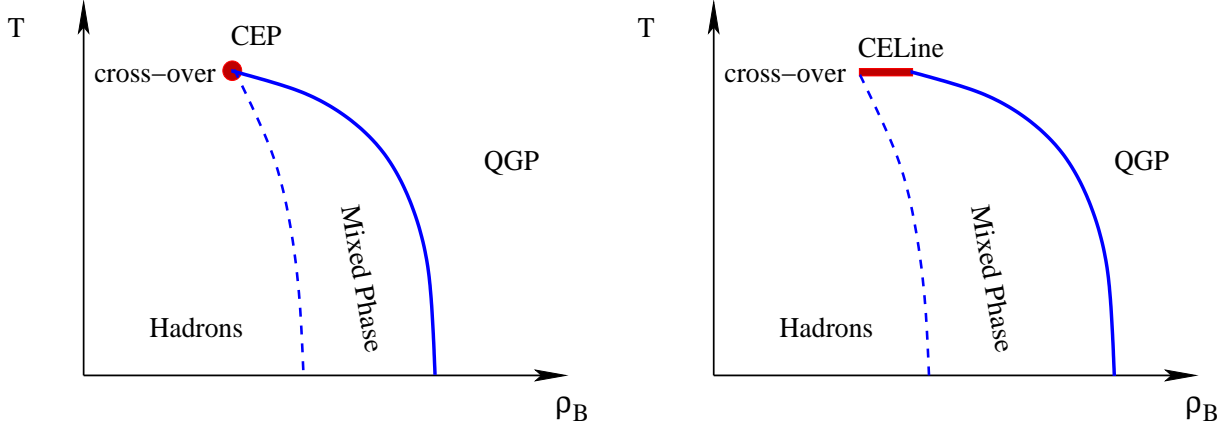


Figure 2: **Left panel.** A schematic picture of the deconfinement phase transition diagram in the plane of baryonic density  $\rho_B$  and  $T$  for the  $2^{nd}$  order PT at the critical endpoint (CEP), i.e. for  $\frac{3}{2} < \tau \leq 2$ . For the  $3^{rd}$  (or higher) order PT the boundary of the mixed and hadronic phases (dashed curve) should have the same slope as the boundary of the mixed phase and QGP (solid curve) at the CEP.

**Right panel.** Same as in the left panel, but for  $\tau > 2$ . The critical endpoint in the  $\mu_B - T$  plane generates the critical end line (CELLine) in the  $\rho_B - T$  plane shown by the thick horizontal line. This occurs because of the discontinuity of the partial derivatives of  $s_H$  and  $s_Q$  functions with respect to  $\mu_B$  and  $T$ .

It is easy to see that the inequalities (11) and (12) are the sufficient conditions of a PT existence for more complicated functional dependencies of  $F_H(s, T, \mu_B)$  and  $F_Q(s, T, \mu_B)$  than the ones used here.

For our choice (9), (10) of  $F_H(s, T, \mu_B)$  and  $F_Q(s, T, \mu_B)$  functions the PT exists at  $T < T_{cep}$ , because the sufficient conditions (11) and (12) can be easily fulfilled by a proper choice of the bag constant  $B$  and the function  $u(T, \mu_B) > 0$  for the interval  $T \leq T_{up}$  with the constant  $T_{up} > T_{cep}$ . Clearly, this is the  $1^{st}$  order PT, since the surface tension is finite and it provides the convergence of the integrals (7) and (8) in the expression (6), where the usual  $T$ -derivatives should be now understood as the partial ones for  $\mu_B = const$ .

Assuming that the conditions (11) and (12) are fulfilled by the correct choice of the model parameters  $B$  and  $u(T, \mu_B) > 0$ , one can see now that at  $T = T_{cep}$  there exists a PT as well, but its order is defined by the value of  $\tau$ . As was discussed in the preceding section for  $\frac{3}{2} < \tau \leq 2$  there exists the  $2^{nd}$  order PT. For  $1 < \tau \leq \frac{3}{2}$  there exist the PT of higher order, defined by the conditions formulated in [25]. This is a new possibility, which, to our best knowledge, does not contradict to any general physical principle (see the left panel in Fig. 2).

The case  $\tau > 2$  can be ruled out because there must exist the first order PT for  $T \geq T_{cep}$ , whereas for  $T < T_{cep}$  there exists the cross-over. Thus, the critical endpoint in  $T - \mu_B$  plane will correspond to the critical interval in the temperature-baryonic density plane. Since such a structure of the phase diagram in the variables temperature-density has, to our knowledge, never been observed, we conclude that the case  $\tau > 2$  is unrealistic (see the right panel in Fig. 2). Note that a similar phase diagram exists in the FDM with the only difference that the boundary of the mixed and liquid phases (the latter in the QGBST model corresponds to QGP) is moved to infinite particle density.

## 4 Surface Tension Induced Phase Transition

Using our results for the case (III) of the preceding section, we conclude that above  $T_{cep}$  there is a cross-over, i.e. the QGP and hadrons coexist together up to the infinite values of  $T$  and/or  $\mu_B$ . Now, however, it is necessary to answer the question: How can the two different sets of singularities that exist on two sides of the line  $T = T_{cep}$  provide the continuity of the solution of Eq. (3)?

It is easy to answer this question for  $\mu_B < \mu_B^c(T_{cep})$  because in this case all partial  $T$  derivatives of  $s_H(T, \mu_B)$ , which is the rightmost singularity, exist and are finite at any point of the line  $T = T_{cep}$ . This can be seen from the fact that for the considered region of parameters  $s_H(T, \mu_B)$  is the rightmost singularity and, consequently,  $s_H(T, \mu_B) > s_Q(T, \mu_B)$ . The latter inequality provides the existence and finiteness of the volume integral in  $F_Q(s, T, \mu_B)$ . In combination with the power  $T$  dependence of the reduced surface tension coefficient  $\sigma(T)$  the same inequality provides the existence and finiteness of all its partial  $T$  derivatives of  $F_Q(s, T, \mu_B)$  regardless to the sign of  $\sigma(T)$ . Thus, using the Taylor expansion in powers of  $(T - T_{cep})$  at any point of the interval  $T = T_{cep}$  and  $\mu_B < \mu_B^c(T_{cep})$ , one can calculate  $s_H(T, \mu_B)$  for the values of  $T > T_{cep}$  which are inside the convergency radius

of the Taylor expansion.

The other situation is for  $\mu_B \geq \mu_B^c(T_{cep})$  and  $T > T_{cep}$ , namely in this case above the deconfinement PT there must exist a weaker PT induced by the disappearance of the reduced surface tension coefficient. To demonstrate this we have solve Eq. (3) in the limit, when  $T$  approaches the curve  $T = T_{cep}$  from above, i.e. for  $T \rightarrow T_{cep} + 0$ , and study the behavior of  $T$  derivatives of the solution of Eq. (3)  $s^*$  for fixed values of  $\mu_B$ . For this purpose we have to evaluate the integrals  $\mathcal{K}_\tau(\Delta, \gamma^2)$  introduced in Eq. (8). Here the notations  $\Delta \equiv s^* - s_Q(T, \mu_B)$  and  $\gamma^2 \equiv -\sigma(T) > 0$  are introduced for convenience.

To avoid the unpleasant behavior for  $\tau \leq 2$  it is convenient to transform (8) further on by integrating by parts:

$$\mathcal{K}_\tau(\Delta, \gamma^2) \equiv g_\tau(V_0) - \frac{\Delta}{(\tau-1)} \mathcal{K}_{\tau-1}(\Delta, \gamma^2) + \frac{\kappa \gamma^2}{(\tau-1)} \mathcal{K}_{\tau-\kappa}(\Delta, \gamma^2), \quad (14)$$

where the regular function  $g_\tau(V_0)$  is defined as

$$g_\tau(V_0) \equiv \frac{1}{(\tau-1) V_0^{\tau-1}} \exp[-\Delta V_0 + \gamma^2 V_0^\kappa]. \quad (15)$$

For  $\tau - a > 1$  one can change the variable of integration  $v \rightarrow z/\Delta$  and rewrite  $\mathcal{K}_{\tau-a}(\Delta, \gamma^2)$  as

$$\mathcal{K}_{\tau-a}(\Delta, \gamma^2) = \Delta^{\tau-a-1} \int_{V_0 \Delta}^{\infty} dz \frac{\exp[-z + \frac{\gamma^2}{\Delta^\kappa} z^\kappa]}{z^{\tau-a}} \equiv \Delta^{\tau-a-1} \mathcal{K}_{\tau-a}(1, \gamma^2 \Delta^{-\kappa}). \quad (16)$$

This result shows that in the limit  $\gamma \rightarrow 0$ , when the rightmost singularity must approach  $s_Q(T, \mu_B)$  from above, i.e.  $\Delta \rightarrow 0^+$ , the function (16) behaves as  $\mathcal{K}_{\tau-a}(\Delta, \gamma^2) \sim \Delta^{\tau-a-1} + O(\Delta^{\tau-a})$ . This is so because for  $\gamma \rightarrow 0$  the ratio  $\gamma^2 \Delta^{-\kappa}$  cannot go to infinity, otherwise the function  $\mathcal{K}_{\tau-1}(1, \gamma^2 \Delta^{-\kappa})$ , which enters into the right hand side of (14), would diverge exponentially and this makes impossible an existence of the solution of Eq. (3) for  $T = T_{cep}$ . The analysis shows that for  $\gamma \rightarrow 0$  there exist two possibilities: either  $\nu \equiv \gamma^2 \Delta^{-\kappa} \rightarrow Const$  or  $\nu \equiv \gamma^2 \Delta^{-\kappa} \rightarrow 0$ . The most straightforward way to analyze these possibilities for  $\gamma \rightarrow 0$  is to assume the following behavior

$$\Delta = A \gamma^\alpha + O(\gamma^{\alpha+1}), \quad \Rightarrow \quad \frac{\partial \Delta}{\partial T} = \frac{\partial \gamma}{\partial T} [A \alpha \gamma^{\alpha-1} + O(\gamma^\alpha)] \sim \frac{(2k+1)A \alpha \gamma^\alpha}{2(T-T_{cep})}, \quad (17)$$

and find out the  $\alpha$  value by equating the  $T$  derivative of  $\Delta$  with the  $T$  derivative (6).

The analysis shows [25] that for  $\Delta^{2-\tau} \leq \gamma \gamma' \Delta^{1-\kappa}$  one finds

$$\gamma^{\alpha-2} \sim \Delta^{1-\kappa} \Rightarrow \alpha \kappa = 2 \quad \text{for} \quad \tau \leq 1 + \frac{\kappa}{2k+1}. \quad (18)$$

Similarly, for  $\Delta^{2-\tau} \geq \gamma \gamma' \Delta^{1-\kappa}$  one obtains  $\gamma^{\alpha-1} \gamma' \sim \Delta^{2-\tau}$  and, consequently,

$$\alpha = \frac{2}{(\tau-1)(2k+1)} \quad \text{for} \quad \tau \geq 1 + \frac{\kappa}{2k+1}. \quad (19)$$

Summarizing our results for  $\gamma \rightarrow 0$ , we can write the expression for the second derivative of  $\Delta$  as [25]:

$$\frac{\partial^2 \Delta}{\partial T^2} \sim \begin{cases} \left[ \frac{T-T_{cep}}{T_{cep}} \right]^{\frac{2k+1}{\kappa}-2}, & \tau \leq 1 + \frac{\kappa}{2k+1}, \\ \left[ \frac{T-T_{cep}}{T_{cep}} \right]^{\frac{3-2\tau}{\tau-1}}, & \tau \geq 1 + \frac{\kappa}{2k+1}. \end{cases} \quad (20)$$

The last result shows us that, depending on  $\kappa$  and  $k$  values, the second derivatives of  $s^*$  and  $s_Q(T, \mu_B)$  can differ from each other for  $\frac{3}{2} < \tau < 2$  or can be equal for  $1 < \tau \leq \frac{3}{2}$ . In other words, we found that at the line  $T = T_{cep}$  there exists the  $2^{nd}$  order PT for  $\frac{3}{2} < \tau < 2$  and the higher order PT for  $1 < \tau \leq \frac{3}{2}$ , which separates the pure QGP phase from the region of a cross-over, i.e. the mixed states of hadronic and QGP bags. Since it exists at the line of a zero surface tension, this PT will be called the *surface induced PT*. For instance, from (20) it follows that for  $k = 0$  and  $\kappa > \frac{1}{2}$  there is the  $2^{nd}$  order PT, whereas for  $k = 0$  and  $\kappa = \frac{1}{2}$  or for  $k > 0$  and  $\kappa < 1$  there is the  $3^{rd}$  order PT, and so on.

Since the analysis performed in the present section did not include any  $\mu_B$  derivatives of  $\Delta$ , it remains valid for the  $\mu_B$  dependence of the reduced surface tension coefficient, i.e. for  $T_{cep}(\mu_B)$ . Only it is necessary to make a few comments on a possible location of the *surface tension null line*  $T_{cep}(\mu_B)$ . In principle, such a null line

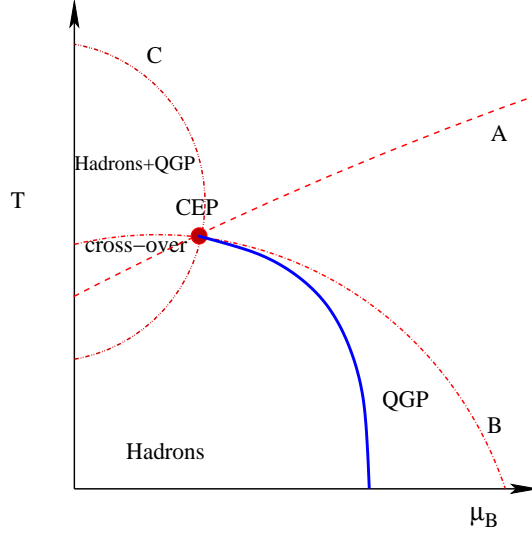


Figure 3: A schematic picture of the deconfinement phase transition diagram (full curve) in the plane of baryonic chemical potential  $\mu_B$  and  $T$  for the  $2^{nd}$  order PT at the tricritical endpoint (CEP). The model predicts an existence of the surface induced PT of the  $2^{nd}$  or higher order (depending on the model parameters). This PT starts at the CEP and goes to higher values of  $T$  and/or  $\mu_B$ . Here it is shown by the dashed curve CEP-A, if the phase diagram is endless, or by the dashed-dot curve CEP-B, if the phase diagram ends at  $T = 0$ , or by the dashed-double-dot curve CEP-C, if the phase diagram ends at  $\mu_B = 0$ . Below (above) each of A or B curves the reduced surface tension coefficient is positive (negative). For the curve C the surface tension coefficient is positive outside of it.

can be located anywhere, if its location does not contradict to the sufficient conditions (11) and (12) of the  $1^{st}$  deconfinement PT existence. Thus, the surface tension null line must cross the deconfinement line in the  $\mu_B - T$  plane at a single point which is the tricritical endpoint  $(\mu_B^{cep}; T_{cep}(\mu_B^{cep}))$ , whereas for  $\mu_B > \mu_B^{cep}$  the null line should have higher temperature for the same  $\mu_B$  than the deconfinement one, i.e.  $T_{cep}(\mu_B) > T_c(\mu_B)$  (see Fig. 3). Clearly, there exist two distinct cases for the surface tension null line: either it is endless, or it ends at zero temperature or at other singularity, like the Color-Flavor-Locked phase. From the present lattice QCD data [38] it follows that the case C in Fig. 3 is the least possible.

To understand the meaning of the surface induced PT it is instructive to quantify the difference between phases by looking into the mean size of the bag:

$$\langle v \rangle \equiv - \left. \frac{\partial \ln F(s, T, \mu_B)}{\partial s} \right|_{s=s^*-0}. \quad (21)$$

As was shown in hadronic phase  $\Delta > 0$  and, hence, it consists of the bags of finite mean volumes, whereas, by construction, the QGP phase is a single infinite bag. For the cross-over states  $\Delta > 0$  and, therefore, they are the bags of finite mean volumes, which gradually increase, if the rightmost singularity approaches  $s_Q(T, \mu_B)$ , i.e. at very large values  $T$  and/or  $\mu_B$ . Such a classification is useful to distinguish QCD phases of present model: it shows that hadronic and cross-over states are separated from the QGP phase by the  $1^{st}$  order deconfinement PT and by the  $2^{nd}$  or higher order PT, respectively.

## 5 Conclusions and Perspectives

Here we discussed an analytically solvable statistical model which simultaneously describes the  $1^{st}$  and  $2^{nd}$  order PTs with a cross-over. The approach is general and can be used for more complicated parameterizations of the hadronic mass-volume spectrum, if in the vicinity of the deconfinement PT region the discrete and continuous parts of this spectrum can be expressed in the form of Eqs. (9) and (10), respectively. Also the actual parameterization of the QGP pressure  $p = Ts_Q(T, \mu_B)$  was not used so far, which means that our result can be extended to more complicated functions, that can contain other phase transformations (chiral PT, or the PT to color superconducting phase) provided that the sufficient conditions (11) and (12) for the deconfinement PT existence are satisfied.

In this model the desired properties of the deconfinement phase diagram are achieved by accounting for the temperature dependent surface tension of the quark-gluon bags. As we showed, it is crucial for the cross-over



existence that at  $T = T_{cep}$  the reduced surface tension coefficient vanishes and remains negative for temperatures above  $T_{cep}$ . Then the deconfinement  $\mu_B - T$  phase diagram has the 1<sup>st</sup> PT at  $\mu_B > \mu_B^c(T_{cep})$  for  $\frac{3}{2} < \tau < 2$ , which degenerates into the 2<sup>nd</sup> order PT (or higher order PT for  $\frac{3}{2} \geq \tau > 1$ ) at  $\mu_B = \mu_B^c(T_{cep})$ , and a cross-over for  $0 \leq \mu_B < \mu_B^c(T_{cep})$ . These two ingredients drastically change the critical properties of the GBM [2] and resolve the long standing problem of a unified description of the 1<sup>st</sup> and 2<sup>nd</sup> order PTs and a cross-over, which, despite all claims, was not resolved in Ref. [7]. In addition, we found that at the null line of the surface tension there must exist the surface induced PT of the 2<sup>nd</sup> or higher order, which separates the pure QGP from the mixed states of hadrons and QGP bags, that coexist above the cross-over region (see Fig. 3). Thus, the QGBST model predicts that the QCD critical endpoint is the tricritical endpoint. It would be interesting to verify this prediction with the help of the lattice QCD analysis. For this one will need to study the behavior of the bulk and surface contributions to the free energy of the QGP bags and/or the string connecting the static quark-antiquark pair.

Also in the QGBST model the pressure of the deconfined phase is generated by the infinite bag, whereas the discrete part of the mass-volume spectrum plays an auxiliary role even above the cross-over region. Therefore, there is no reason to believe that any quantitative changes of the properties of low lying hadronic states generated by the surrounding media (like the mass shift of the  $\omega$  and  $\rho$  mesons [39]) would be the robust signals of the deconfinement PT. On the other hand, the QGP bags created in the experiments have finite mass and volume and, hence, the strong discontinuities which are typical for the 1<sup>st</sup> order PT should be smeared out which would make them hardly distinguishable from the cross-over. Thus, to seriously discuss the signals of the 1<sup>st</sup> order deconfinement PT and/or the tricritical endpoint, one needs to solve the finite volume version of the QGBST model like it was done for the SMM [21] and the GBM [22]. This, however, is not sufficient because, in order to make any reliable prediction for experiments, the finite volume equation of state must be used in hydrodynamic equations which, unfortunately, are not suited for such a purpose. Thus, we are facing a necessity to return to the foundations of heavy ion phenomenology and to modify them according to the requirements of the experiments.

To apply the QGBST model to the experiments it is necessary to refine it: it seems that for the mixture of hadrons and QGP bags above the cross-over line it is necessary to include the relativistic treatment of hard core repulsion [40, 41] for lightest hadrons and to include into statistical description the medium dependent width of resonances and QGP bags, which can, in principal, change our understanding of the cross-over mechanism [42].

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